

RIEMANN COMPATIBLE TENSORS

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ABSTRACT. Derdzinski and Shen’s theorem on the restrictions posed by a Codazzi tensor on the Riemann tensor holds more generally when a Riemann-compatible tensor exists. Several properties are shown to remain valid in this broader setting. Riemann compatibility is equivalent to the Bianchi identity of the new “Codazzi deviation tensor”, with a geometric significance. The general properties are studied, with their implications on Pontryagin forms. Examples are given of manifolds with Riemann-compatible tensors, in particular those generated by geodesic mapping. Compatibility is extended to generalized curvature tensors, with an application to Weyl’s tensor and general relativity.

1. INTRODUCTION

The Riemann tensor $R_{ijk}{}^m$ and its contractions, $R_{kl} = R_{kml}{}^m$ and $R = g^{kl}R_{kl}$, are the fundamental tensors to describe the local structure of a Riemannian manifold (\mathcal{M}_n, g) of dimension n . In a remarkable theorem [10, 3] Derdzinski and Shen showed that the existence of a non trivial Codazzi tensor poses strong constraints on the structure of the Riemann tensor. Because of their geometric relevance, Codazzi tensors have been studied by several authors, as Berger and Ebin [1], Bourguignon [4], Derdzinski [8, 9], Derdzinski and Shen [10], Ferus [11], Simon [29]; a compendium of results is found in Besse’s book [3]. Recently, we showed [22] that the Codazzi differential condition

$$(1) \quad \nabla_i b_{jk} - \nabla_j b_{ik} = 0$$

is sufficient for the theorem to hold, and can be replaced by the more general notion of *Riemann-compatibility*, which is instead algebraic:

Definition 1.1. A symmetric tensor b_{ij} is Riemann compatible (R -compatible) if:

$$(2) \quad b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m = 0.$$

With this requirement, we proved the following extension of Derdzinski-Shen’s theorem:

Theorem 1.2. [22] *Suppose that a symmetric R -compatible tensor b_{ij} exists. Then, if X , Y and Z are three eigenvectors of the matrix $b_r{}^s$ at a point of the manifold, with eigenvalues λ , μ and ν , it is $R_{ijkl}X^iY^jZ^k = 0$ provided that both λ and μ are different from ν .*

The concept of compatibility allows for a further extension of the theorem, where the Riemann tensor R is replaced by a generalized curvature tensor K , and b is

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required to be K -compatible [22].

This paper studies the properties of Riemann compatibility, and its implications on the geometry of the manifold. In section 2 R -compatibility is shown to be equivalent to the Bianchi identity of a new tensor, the *Codazzi deviation*. In section 3 the irreducible components of the covariant derivative of a symmetric tensor are classified in a simple manner, based on the decomposition into traceless terms. This is of guidance in the study of different structures suited for R -compatibility. The general properties of Riemann compatibility are presented in section 4. In section 5 several properties of manifolds in presence of a Riemann compatible tensor that were obtained by Derdzinsky-Shen and Bourguignon for manifolds with a Codazzi tensor, are recovered. In particular, it is shown that R -compatibility implies pureness, a property of the Riemann tensor introduced by Maillot that implies the vanishing of Pontryagin forms. Manifolds that display R -compatible tensors are presented in section 6; interesting examples are generated by geodesic mappings, that induce metric tensors that are R -compatible. Finally, in section 7, K -tensors and K -compatibility are presented, with applications to the standard curvature tensors. In the end, an application to general relativity is mentioned, that will be discussed fully elsewhere.

2. THE CODAZZI DEVIATION TENSOR AND R -COMPATIBILITY

Since Codazzi tensors are Riemann compatible, for a non Codazzi differentiable symmetric tensor field b it is useful to define its deviation from the Codazzi condition. This tensor solves an unexpected relation that generalizes Lovelock's identity for the Riemann tensor, and shows that Riemann compatibility is a condition for closedness of certain 2-forms.

Definition 2.1. The *Codazzi deviation* of a symmetric tensor b_{kl} is

$$(3) \quad \mathcal{C}_{jkl} =: \nabla_j b_{kl} - \nabla_k b_{jl}$$

Simple properties are: $\mathcal{C}_{jkl} = -\mathcal{C}_{kjl}$ and $\mathcal{C}_{jkl} + \mathcal{C}_{klj} + \mathcal{C}_{ljk} = 0$.

The following identity holds in general, and relates the Bianchi differential combination for \mathcal{C} to the Riemann compatibility of b :

Proposition 2.2.

$$(4) \quad \nabla_i \mathcal{C}_{jkl} + \nabla_j \mathcal{C}_{kil} + \nabla_k \mathcal{C}_{ijl} = b_{im} R_{jkl}^m + b_{jm} R_{kil}^m + b_{km} R_{ijl}^m$$

Proof.

$$\begin{aligned} \nabla_i \mathcal{C}_{jkl} + \nabla_j \mathcal{C}_{kil} + \nabla_k \mathcal{C}_{ijl} &= [\nabla_i, \nabla_j] b_{kl} + [\nabla_k, \nabla_i] b_{jl} + [\nabla_j, \nabla_k] b_{il} \\ &= b_{ml} (R_{ijk}^m + R_{kij}^m + R_{jki}^m) + b_{im} R_{jkl}^m + b_{jm} R_{kil}^m + b_{km} R_{ijl}^m \end{aligned}$$

the first term vanishes by the first Bianchi identity. \square

Remark 2.3. The identity holds true if b_{ij} is replaced by $b'_{ij} = b_{ij} + \chi a_{ij}$, where a_{ij} is a Codazzi tensor and χ a scalar field. Then: $\mathcal{C}'_{jkl} = \mathcal{C}_{jkl} - (a_{kl} \nabla_j - a_{jl} \nabla_k) \chi$.

The deviation tensor is associated to the 2-form $\mathcal{C}_l = \frac{1}{2} \mathcal{C}_{jkl} dx^j \wedge dx^k$. The closedness condition $0 = D\mathcal{C}_l = \frac{1}{2} \nabla_i \mathcal{C}_{jkl} dx^i \wedge dx^j \wedge dx^k$ (D is the exterior covariant derivative) is the second Bianchi identity for the Codazzi deviation: $\nabla_i \mathcal{C}_{jkl} + \nabla_j \mathcal{C}_{kil} + \nabla_k \mathcal{C}_{ijl} = 0$. This gives a geometric picture of Riemann compatibility:

Theorem 2.4. b_{ij} is Riemann compatible if and only if $D\mathcal{C}_l = 0$.

Remark 2.5. The Codazzi deviation of the Ricci tensor is, by the contracted second Bianchi identity: $\mathcal{C}_{jkl} =: \nabla_j R_{kl} - \nabla_k R_{jl} = -\nabla_m R_{jkl}{}^m$. For the Ricci tensor the identity (4) identifies with Lovelock's identity [18] for the Riemann tensor:

$$(5) \quad \begin{aligned} & \nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kil}{}^m + \nabla_k \nabla_m R_{ijl}{}^m \\ & = -R_{im} R_{jkl}{}^m - R_{jm} R_{kil}{}^m - R_{km} R_{ijl}{}^m. \end{aligned}$$

A Veblen-like identity holds, that corresponds to (14) (For $b_{ij} = R_{ij}$ it specializes to Veblen's identity for the divergence of the Riemann tensor [20]):

Proposition 2.6.

$$(6) \quad \begin{aligned} & \nabla_i \mathcal{C}_{jlk} + \nabla_j \mathcal{C}_{kil} + \nabla_k \mathcal{C}_{lji} + \nabla_l \mathcal{C}_{ikj} \\ & = b_{im} R_{jlk}{}^m + b_{jm} R_{kil}{}^m + b_{km} R_{lji}{}^m + b_{lm} R_{ikj}{}^m \end{aligned}$$

Proof. Write four equations (4) with cycled indices i, j, k, l and sum up. Then simplify by means of the first Bianchi identity for the Riemann tensor and the cyclic identity $\mathcal{C}_{jkl} + \mathcal{C}_{klj} + \mathcal{C}_{ljk} = 0$. \square

3. IRREDUCIBLE COMPONENTS FOR $\nabla_j b_{kl}$ AND R -COMPATIBILITY

We begin with a simple procedure to classify the $O(n)$ invariant components of the tensor $\nabla_j b_{kl}$. They will guide us in the study of R -compatibility. If b is the Ricci tensor, this simple construction reproduces the seven equations linear in $\nabla_i R_{jk}$, invariant for the $O(n)$ group, that are enumerated and discussed in Besse's treatise "Einstein Manifolds" [3].

For a symmetric tensor b_{kl} with $\nabla_j b_{kl} \neq 0$, the tensor $\nabla_j b_{kl}$ can be decomposed into $O(n)$ invariant terms, where \mathcal{B}_{jkl}^0 is traceless ($\mathcal{B}_{jk}^0{}^j = \mathcal{B}_{kj}^0{}^j = 0$) [14, 17]:

$$(7) \quad \nabla_j b_{kl} = \mathcal{B}_{jkl}^0 + A_j g_{kl} + B_k g_{jl} + B_l g_{jk}$$

$$(8) \quad A_j = \frac{(n+1)\nabla_j b^m{}_m - 2\nabla_m b^m{}_j}{n^2 + n - 2}, \quad B_j = -\frac{\nabla_j b^m{}_m - n\nabla_m b^m{}_j}{n^2 + n - 2}$$

The traceless tensor can then be written as a sum of orthogonal components [18]:

$$(9) \quad \mathcal{B}_{jkl}^0 = \frac{1}{3} [\mathcal{B}_{jkl}^0 + \mathcal{B}_{klj}^0 + \mathcal{B}_{ljk}^0] + \frac{1}{3} [\mathcal{B}_{jkl}^0 - \mathcal{B}_{kjl}^0] + \frac{1}{3} [\mathcal{B}_{jlk}^0 - \mathcal{B}_{ljk}^0]$$

The orthogonal subspaces classify the $O(n)$ invariant equations that are linear in $\nabla_j b_{kl}$. The trivial subspace: $\nabla_j b_{kl} = 0$. The subspace \mathcal{I} (we follow Gray's notation, [13]) where $\mathcal{B}_{jkl}^0 = 0$:

$$\nabla_j b_{kl} = A_j g_{kl} + B_k g_{jl} + B_l g_{jk}.$$

The complement \mathcal{I}^\perp is characterized by $A_j, B_j = 0$ i.e. $\nabla_j b_{kl}$ is traceless. This gives two invariant equations: $\nabla_j b^j{}_l = 0$, and $\nabla_j b^m{}_m = 0$. Since $\nabla_j b_{kl} = \mathcal{B}_{jkl}^0$, the structure of \mathcal{B}^0 specifies two orthogonal subspaces $\mathcal{I}^\perp = \mathcal{A} \oplus \mathcal{B}$. In \mathcal{A} :

$$\nabla_j b_{kl} + \nabla_k b_{lj} + \nabla_l b_{jk} = 0.$$

In \mathcal{B} :

$$\nabla_j b_{kl} - \nabla_k b_{jl} = 0.$$

The subspace $\mathcal{I} \oplus \mathcal{A}$ contains tensors with traceless part $\nabla_j b_{kl} - A_j g_{kl} - B_k g_{jl} - B_l g_{jk}$ that solves the cyclic condition:

$$[\nabla_j b_{kl} - \frac{1}{n+2}(\nabla_j b^m{}_m + 2\nabla_m b^m{}_j)g_{kl}] + cyclic = 0.$$

The subspace $\mathcal{I} \oplus \mathcal{B}$ contains tensors with traceless part that solves the Codazzi condition:

$$[\nabla_j b_{kl} - \frac{1}{n-1}(\nabla_j b^m{}_m - \nabla_m b^m{}_j)g_{kl}] = [\nabla_k b_{jl} - \frac{1}{n-1}(\nabla_k b^m{}_m - \nabla_m b^m{}_k)g_{jl}]$$

Accordingly, the Codazzi deviation tensor has the (unique) decomposition in irreducible components

$$(10) \quad \mathcal{C}_{jkl} = \mathcal{C}_{jkl}^0 + \lambda_j g_{kl} - \lambda_k g_{jl}, \quad \lambda_j = A_j - B_j = \frac{\nabla_j b^m{}_m - \nabla_m b^m{}_j}{n-1}$$

where \mathcal{C}^0 is traceless. Eq.(4) becomes

$$(11) \quad b_{im} R_{jkl}{}^m + b_{jm} R_{kil}{}^m + b_{km} R_{ijl}{}^m = \nabla_i \mathcal{C}_{jkl}^0 + \nabla_j \mathcal{C}_{kil}^0 + \nabla_k \mathcal{C}_{ijl}^0 \\ + g_{il}(\nabla_j \lambda_k - \nabla_k \lambda_j) + g_{jl}(\nabla_k \lambda_i - \nabla_i \lambda_k) + g_{kl}(\nabla_i \lambda_j - \nabla_j \lambda_i)$$

There are only two orthogonal invariant cases:

- $\mathcal{C}_{jkl}^0 = 0$, then b is R -compatible if and only if λ is closed. If b is the Ricci tensor, this requirement gives Nearly conformally symmetric $(NCS)_n$ manifolds, that were introduced by Roter [28].

- $\nabla_j b^m{}_m - \nabla_m b^m{}_j = 0$ then b is R -compatible if and only if $\mathcal{C} = \mathcal{C}^0$ solves the second Bianchi identity. If b is the Ricci tensor, this corresponds to $\nabla_j R = 0$.

Remark 3.1. The decomposition (10) for the deviation of the Ricci tensor turns out to be

$$(12) \quad \mathcal{C}_{jkl} = -\frac{n-2}{n-3} \nabla_m C_{jkl}{}^m + \frac{1}{2(n-1)} [g_{kl} \nabla_j R - g_{jl} \nabla_k R]$$

where $C_{jkl}{}^m$ is the conformal curvature tensor, or Weyl's tensor. In this case the λ covector is closed.

4. RIEMANN COMPATIBILITY: GENERAL PROPERTIES

The existence of a Riemann compatible tensor has various implications. A first one is the existence of a generalized curvature tensor. This leads to the generalization of Derdzinski-Shen theorem and other relations that were obtained for Codazzi tensors.

We need the definition, from Kobayashi and Nomizu's book [16]:

Definition 4.1. A tensor K_{ijklm} is a *generalized curvature tensor* (or, briefly, a K -tensor) if it has the symmetries of the Riemann curvature tensor:

- a) $K_{ijkl} = -K_{jikl} = -K_{ijlk}$,
- b) $K_{ijkl} = K_{klij}$,
- c) $K_{ijkl} + K_{jkil} + K_{kijl} = 0$ (first Bianchi identity).

It follows that the tensor $K_{jk} =: -K_{mjk}{}^m$ is symmetric.

Theorem 4.2. If b is R -compatible then $K_{ijkl} =: R_{ijpq} b^p{}_k b^q{}_l$ is a K -tensor.

Proof. a) For example: $K_{ijlk} = R_{ijrs}b_l^r b_k^s = R_{ijsr}b_l^s b_k^r = -R_{ijrs}b_l^s b_k^r = -K_{ijkl}$. Property c) follows from (2): $K_{ijkl} + K_{jkil} + K_{kijl} = R_{ijrs}b_k^r b_l^s + R_{jkr s}b_i^r b_l^s + R_{kirs}b_j^r b_l^s = (R_{jis}^r b_{kr} + R_{kjs}^r b_{ir} + R_{iks}^r b_{jr})b_l^s = 0$. Property b) follows from c): $K_{ijkl} + K_{jkil} + K_{kijl} = 0$. Sum the identity over cyclic permutations of all indices i, j, k, l and use the symmetries a). It is easy to see that a first Bianchi identity holds also for the last three indices: $K_{ijkl} + K_{iklj} + K_{iljk} = 0$. \square

The next result remarks the relevance of the local basis of eigenvectors of the Ricci tensor. Another symmetric contraction of the Riemann tensor was introduced by Bourguignon [4]:

$$(13) \quad \mathring{R}_{ij} =: b^{pq} R_{pijq}.$$

Theorem 4.3. *If b is R -compatible then:*

- 1) $b_{im} R_j^m - b_{jm} R_i^m = 0$,
- 2) $b_{im} \mathring{R}_j^m - b_{jm} \mathring{R}_i^m = 0$

Proof. The first identity is proven by transvecting (2) with g^{kl} . The second one is a restatement of the symmetry of the tensor K_{ij} . \square

Remark 4.4. A) Identities 1 and 2 are here obtained directly from R -compatibility. Bourguignon [4] obtained them from Weitzenböck's formula for Codazzi tensors, and Derdzinski and Shen [10] from their theorem.

B) As the symmetric matrices b_{ij} , R_{ij} , \mathring{R}_{ij} commute, they share at each point of the manifold an orthonormal set of n eigenvectors.

C) If b' is a symmetric tensor that commutes with a Riemann compatible b , then it can be shown that $\mathring{R}'_{ij} =: b'^{pq} R_{pijq}$ commutes with b .

Finally, this Veblen-type identity holds:

Proposition 4.5. *If b is R -compatible, then:*

$$(14) \quad b_{im} R_{jlk}^m + b_{jm} R_{kil}^m + b_{km} R_{lji}^m + b_{lm} R_{ikj}^m = 0$$

Proof. Write four equations (2) with cycled indices i, j, k, l and sum up, and use the first Bianchi identity. \square

5. PURE RIEMANN TENSORS AND PONTRYAGIN FORMS

Riemann compatibility and nondegeneracy of the eigenvalues of b imply directly that the Riemann tensor is *pure* and Pontryagin forms vanish.

We quote two results from Maillot's paper [19]:

Definition 5.1. In a Riemann manifold \mathcal{M}_n , the Riemann curvature tensor is pure if at each point of the manifold there is an orthonormal basis of n tangent vectors $X(1), \dots, X(n)$, $X(a)^i X(b)_i = \delta_{ab}$, such that the tensors $X(a)^i \wedge X(b)^j =: X(a)^i X(b)^j - X(a)^j X(b)^i$, $a < b$, diagonalize it:

$$(15) \quad R_{ij}{}^{lm} X(a)^i \wedge X(b)^j = \lambda_{ab} X(a)^l \wedge X(b)^m$$

Theorem 5.2. *If a Riemannian manifold has pure Riemann curvature tensor, then all Pontryagin forms vanish.*

Consider the maps on tangent vectors, built with the Riemann tensor,

$$\begin{aligned}\omega_4(X_1 \dots X_4) &= R_{ija}{}^b R_{klb}{}^a (X_1^i \wedge X_2^j)(X_3^k \wedge X_4^l), \\ \omega_8(X_1 \dots X_8) &= R_{ija}{}^b R_{klb}{}^c R_{mnc}{}^d R_{pqd}{}^a (X_1^i \wedge X_2^j) \dots (X_7^p \wedge X_8^q), \\ &\dots\dots\end{aligned}$$

They are antisymmetric under exchange of vectors in the single pairs, and for cyclic permutation of pairs. The *Pontryagin forms* [26] Ω_{4k} result from total antisymmetrization of ω_{4k} : $\Omega_{4k}(X_1 \dots X_{4k}) = \sum_P (-1)^P \omega_{4k}(X_{i_1} \dots X_{i_{4k}})$ where P is the permutation taking $(1 \dots 4k)$ to $(i_1 \dots i_{4k})$. $\Omega_{4k} = 0$ if two vectors repeat, intermediate forms Ω_{4k-2} vanish identically.

Pontryagin forms on generic tangent vectors are linear combinations of forms evaluated on basis vectors.

If the Riemann tensor is pure, all Pontryagin forms on the basis of eigenvectors of the Riemann tensor vanish. For example, if X, Y, Z, W are orthogonal: $\omega_4(XYZW) = \lambda_{XY}\lambda_{ZW}(X^a \wedge Y^b)(Z_b \wedge W_a) = 0$ and $\Omega_4(XYZU) = 0$.

A consequence of the extended Derdzinski-Shen theorem 1.2 is the following:

Theorem 5.3. *If a symmetric tensor field b_{ij} exists, that is R -compatible and has distinct eigenvalues at each point of the manifold, then the Riemann tensor is pure and all Pontryagin forms vanish.*

Proof. At each point of the manifold the symmetric matrix $b_{ij}(x)$ is diagonalized by n tangent orthonormal vectors $X(a)$, with distinct eigenvalues. Since b is R -compatible, theorem 1.2 holds and, because of antisymmetry of R in first two indices:

$$0 = R_{ij}{}^{kl} X(a)^i \wedge X(b)^j X(c)_k, \quad a \neq b \neq c.$$

This means that all column vectors of the matrix $V(a, b)^{kl} = R_{ij}{}^{kl} X(a)^i \wedge X(b)^j$ are orthogonal to vectors $X(c)$ i.e. they belong to the subspace spanned by $X(a)$ and $X(b)$. Because of antisymmetry in indices k, l , it is necessarily $V(a, b) = \lambda_{ab} X(a) \wedge X(b)$, i.e. the Riemann tensor is pure. \square

This property has been checked by Petersen [27] in various examples with rotationally invariant metrics, by giving explicit orthonormal frames such that $R(e_i, e_j)e_k = 0$.

5.1. Two and three dimensional manifolds. Riemannian manifolds of dimension $n = 2$ and $n = 3$ are special, as the Riemann tensor is expressible in terms of the Ricci and metric tensors. Therefore, Riemann-compatibility and ensuing pureness of the Riemann tensor can be established by simple means.

$n = 2$) $R_{jklm} = R_{jl}g_{km} - g_{jm}R_{kj}$. Explicit evaluation proves that any symmetric tensor b is Riemann compatible.

$n = 3$) $R_{jklm} = g_{jl}R_{km} + g_{km}R_{jl} - g_{kl}R_{jm} - g_{jm}R_{kl} - \frac{R}{2}(g_{jl}g_{km} - g_{jm}g_{kl})$. Then, for any symmetric tensor b it is:

$$\begin{aligned}b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m &= g_{kl}(b_{jm}R_i{}^m - b_{im}R_j{}^m) \\ &+ g_{il}(b_{km}R_j{}^m - b_{jm}R_k{}^m) + g_{jl}(b_{im}R_k{}^m - b_{km}R_i{}^m)\end{aligned}$$

Thus in $n = 3$ the Ricci tensor is always R -compatible. Moreover, if b commutes with the Ricci tensor, then b is R -compatible. Since a symmetric tensor that commutes with the Ricci tensor can always be constructed, with arbitrarily chosen distinct eigenvalues, by theorem 5.3 we conclude:

Proposition 5.4. *In Riemannian manifolds of dimension $n = 2$ and $n = 3$ the Riemann tensor is pure.*

5.2. Quasi-constant curvature spaces. The same conclusions can be drawn in any dimension n for quasi-constant curvature spaces. They were introduced by Chen and Yano [5] and are defined by a Riemann tensor with the following structure:

$$(16) \quad R_{jklm} = p[g_{mj}g_{kl} - g_{mk}g_{jl}] + q[g_{mj}t_k t_l - g_{mk}t_j t_l + g_{kl}t_m t_j - g_{jl}t_m t_k]$$

p and q are scalar functions. The first term describes constant curvature, the second one contains a vector field with $t_k t^k = 1$.

The following identity holds:

$$(17) \quad b_i^m R_{jklm} + b_j^m R_{kil m} + b_k^m R_{ijl m} = q[g_{kl}(t_j b_i^m t_m - t_i b_j^m t_m) + g_{il}(t_k b_j^m t_m - t_j b_k^m t_m) + g_{jl}(t_i b_k^m t_m - t_k b_i^m t_m)]$$

Contraction with g^{kl} gives: $-b_i^m R_{jm} + b_j^m R_{im} = q(n-2)(t_j b_i^m t_m - t_i b_j^m t_m)$. Therefore, if b commutes with the Ricci tensor and $n \neq 2$, the r.h.s. is zero and, by (17), b is R -compatible. Then the Riemann tensor is pure and all Pontryagin forms vanish.

6. STRUCTURES FOR RIEMANN COMPATIBILITY

Some differential structures are presented that yield Riemann compatibility. Of particular interest are geodesic mappings, which leave the condition for R -compatibility form-invariant, and generate R -compatible tensors.

6.1. Quasi Codazzi tensors. Let b_{ij} be a symmetric tensor that solves the Codazzi condition deformed by a closed gauge field [22]:

$$(18) \quad (\nabla_j - \beta_j)b_{kl} = (\nabla_k - \beta_k)b_{jl}$$

The Codazzi deviation is $\mathcal{C}_{jkl} = \beta_j b_{kl} - \beta_k b_{jl}$, and b is R -compatible.

Since $\beta_i = \nabla_i \xi$, the gauge field cancels for $b_{ij} = e^\xi b'_{ij}$, where b' is a Codazzi tensor.

Of this type are Weakly b -symmetric manifolds, defined by the recurrency

$$(19) \quad \nabla_i b_{kl} = A_i b_{kl} + B_k b_{il} + D_l b_{ik}$$

where A , B and D are covector fields. Eq.(18) is obtained for $\beta_i = A_i - B_i$, and b is Riemann compatible if $A - B$ is closed.

Examples are: Weakly Ricci-symmetric manifolds, where $b_{ij} = R_{ij}$ [20, 21], Weakly and pseudo Z -symmetric manifolds, where b_{ij} is a Z -tensor [21, 23]. Another example are manifolds with a recurrent generalized curvature tensor [20]: $\nabla_i K_{jkl}^m = A_i K_{jkl}^m$, then $b_{kl} =: K_{kml}^m \neq 0$ has the form (19).

6.2. Pseudo- K -symmetric manifolds. They are characterized by a generalized curvature tensor K such that ([6, 24])

$$\nabla_i K_{jkl}^m = 2A_i K_{jkl}^m + A_j K_{ikl}^m + A_k K_{jil}^m + A_l K_{jki}^m + A^m K_{jkli},$$

The tensor $b_{jk} =: K_{jmk}^m$ is symmetric. It is R -compatible if its Codazzi deviation $\mathcal{C}_{ikl} = A_i b_{kl} - A_k b_{il} + 3A_m K_{ikl}^m$ fulfills the II Bianchi identity. This is ensured by A_m being concircular, i.e. $\nabla_i A_m = A_i A_m + \gamma g_{im}$.

6.3. Generalized Weyl tensors. A Riemannian manifold is a $(NCS)_n$ [28] if the Ricci tensor satisfies $\nabla_j R_{kl} - \nabla_k R_{jl} = \frac{1}{2(n-1)}[g_{kl}\nabla_j R - g_{jl}\nabla_k R]$. As such, the Ricci tensor is the Weyl tensor, and the left hand side is its Codazzi deviation. This condition, by (12), is equivalent to $\nabla_m C_{jkl}^m = 0$.

This suggests a class of deviations of a symmetric tensor b with $\mathcal{C}_{jkl}^0 = 0$ in (10):

$$(20) \quad \mathcal{C}_{jkl} = \lambda_j g_{kl} - \lambda_k g_{jl}$$

Proposition 6.1. *b is R -compatible if and only if λ_i is closed.*

Proof. Transvect (11) with g^{kl} and obtain: $-b_i^m R_{jm} + b_j^m R_{im} = (n-2)(\nabla_i \lambda_j - \nabla_j \lambda_i)$. Then b commutes with the Ricci tensor iff λ is closed and, by the previous equation, b is R -compatible. \square

An example is provided by spaces with

$$(21) \quad \nabla_j b_{kl} = A_j g_{kl} + B_k g_{jl} + B_l g_{jk},$$

where $\mathcal{C}_{jkl} = \lambda_j g_{kl} - \lambda_k g_{jl}$ with $\lambda_j = A_j - B_j$. Sinyukov manifolds [30] are of this sort, with b_{ij} being the Ricci tensor itself.

6.4. Geodesic mappings. Riemann compatible tensors arise naturally in the study of geodesic mappings, i.e. mappings that preserve geodesic lines. Their importance arise from the fact that Sinyukov manifolds are $(NCS)_n$ manifolds and they always admit a nontrivial geodesic mapping.

Geodesic mappings preserve Weyl's projective curvature tensor [30]. We show that they also preserve the form of the compatibility relation.

A map $f : (\mathcal{M}_n, g) \rightarrow (\mathcal{M}_n, \bar{g})$ is *geodesic* if and only if Christoffel symbols are linked by $\bar{\Gamma}_{ij}^k = \Gamma_{ij}^k + \delta_i^k X_j + \delta_j^k X_i$ where, on a Riemannian manifold, X is closed ($\nabla_i X_j = \nabla_j X_i$). The condition is equivalent to:

$$(22) \quad \nabla_k \bar{g}_{jl} = 2X_k \bar{g}_{jl} + X_j \bar{g}_{kl} + X_l \bar{g}_{kj}$$

which has the form (21). The corresponding relation among Riemann tensors is

$$(23) \quad \bar{R}_{jkl}^m = R_{jkl}^m + \delta_j^m P_{kl} - \delta_k^m P_{jl}$$

where $P_{kl} = \nabla_k X_l - X_k X_l$ is the *deformation* tensor. The symmetry $P_{kl} = P_{lk}$ is ensured by closedness of X .

Proposition 6.2. *Geodesic mappings preserve R -compatibility*

$$(24) \quad b_{im} \bar{R}_{jkl}^m + b_{jm} \bar{R}_{kil}^m + b_{km} \bar{R}_{ijl}^m = b_{im} R_{jkl}^m + b_{jm} R_{kil}^m + b_{km} R_{ijl}^m$$

where b is a symmetric tensor.

Proof. Let's show that the difference of the two sides is zero. Eq.(23) gives:

$$\begin{aligned} & b_{im}(\delta_j^m P_{kl} - \delta_k^m P_{jl}) + b_{jm}(\delta_k^m P_{il} - \delta_i^m P_{kl}) + b_{km}(\delta_i^m P_{jl} - \delta_j^m P_{il}) \\ &= b_{ij} P_{kl} - b_{ik} P_{jl} + b_{jk} P_{il} - b_{ji} P_{kl} + b_{ki} P_{jl} - b_{kj} P_{il} = 0 \end{aligned} \quad \square$$

Since \bar{g} is trivially \bar{R} -compatible (first Bianchi identity), form invariance implies:

Corollary 6.3. *\bar{g} is R -compatible.*

Sinyukov [30] (see also [25, 12]) showed that a manifold admits a geodesic mapping if and only if there are a scalar field φ and a symmetric non singular tensor b_{ij} such that:

$$\nabla_k b_{jl} = g_{kl} \nabla_j \varphi + g_{kj} \nabla_l \varphi.$$

The Codazzi deviation of b , $\mathcal{C}_{jkl} = g_{kl}\nabla_j\varphi - g_{jl}\nabla_k\varphi$, has the form (20). Therefore b is R -compatible.

7. GENERALIZED CURVATURE TENSORS.

Several results that are valid for the Riemann tensor with a Riemann compatible tensor, extend to generalized curvature tensors K_{ijkl} (hereafter referred to as K -tensors) with a K -compatible symmetric tensor b_{jk} . The classical curvature tensors are K -tensors. The compatibility with the Ricci tensor is then examined.

Definition 7.1. A symmetric tensor b_{ij} is K -compatible if

$$(25) \quad b_{im}K_{jkl}{}^m + b_{jm}K_{kil}{}^m + b_{km}K_{ijl}{}^m = 0.$$

The metric tensor is always K -compatible, as (25) then coincides with the first Bianchi identity for K .

Proposition 7.2. If K_{ijlm} is a K -tensor and b_{kl} is K -compatible, then $\hat{K}_{ijkl} =: K_{ijrs}b_k{}^rb_l{}^s$ is a K -tensor.

We quote without proof the extension of Derdzinski and Shen theorem for generalized curvature tensors [22]:

Theorem 7.3. Suppose that K_{ijkl} is a K -tensor, and a symmetric K -compatible tensor b_{ij} exists. Then, if X, Y and Z are three eigenvectors of the matrix $b_r{}^s$ at a point x of the manifold, with eigenvalues λ, μ and ν , it is $X^iY^jZ^kK_{ijkl} = 0$ provided that both λ and μ are different from ν .

Proposition 7.4. If b is K -compatible, and b commutes with a tensor h , then the symmetric tensor $\hat{K}_{kl} =: K_{jklm}h^j{}^m$ commutes with b .

Proof. Multiply relation of K compatibility for b by h^{kl} . The last term vanishes for symmetry. The remaining terms give the null commutation relation. \square

In ref.[20] (prop.2.4) we proved that a generalization of Lovelock's identity (5) holds for certain K -tensors, that include all classical curvature tensors:

Proposition 7.5. Let $K_{jkl}{}^m$ be a K -tensor with the property

$$(26) \quad \nabla_m K_{jkl}{}^m = \alpha \nabla_m R_{jkl}{}^m + \beta (a_{kl}\nabla_j - a_{jl}\nabla_k)\varphi,$$

where α, β are non zero constants, φ is a real scalar function and a_{kl} is a Codazzi tensor. Then:

$$(27) \quad \begin{aligned} & \nabla_i \nabla_m K_{jkl}{}^m + \nabla_j \nabla_m K_{kil}{}^m + \nabla_k \nabla_m K_{ijl}{}^m \\ & = -\alpha (R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m). \end{aligned}$$

7.1. ABC curvature tensors. A class of K -tensors with the structure (26) are the ABC curvature tensors. They are combinations of the Riemann tensor and its contractions (A, B, C are constants unless otherwise stated):

$$(28) \quad \begin{aligned} K_{jkl}{}^m = & R_{jkl}{}^m + A(\delta_j{}^m R_{kl} - \delta_k{}^m R_{jl}) + B(R_j{}^m g_{kl} - R_k{}^m g_{jl}) \\ & + C(R\delta_j{}^m g_{kl} - R\delta_k{}^m g_{jl}) \end{aligned}$$

The canonical curvature tensors are of this sort:

- *Conformal* tensor C_{ijkl} : $A = B = \frac{1}{n-2}$, $C = -\frac{1}{(n-1)(n-2)}$;

- *Conharmonic* tensor N_{ijkl} : $A = B = \frac{1}{n-2}$, $C = 0$;
- *Projective* tensor: P_{ijkl} : $A = \frac{1}{n-1}$, $B = C = 0$;
- *Concircular* tensor: \tilde{C}_{ijkl} : $A - B = 0$, $C = \frac{1}{n(n-1)}$.

Proposition 7.6. *Let K_{ijkl} be an ABC tensor (A, B, C may be scalar functions) and b_{ij} a symmetric tensor;*

- 1) *if b is R -compatible then b is K -compatible.*
- 2) *if b is K -compatible and $B \neq \frac{1}{n-2}$ then b is R -compatible.*

Proof. The following identity holds for ABC tensors and a symmetric tensor b :

$$(29) \quad b_{im}K_{jkl}{}^m + b_{jm}K_{kil}{}^m + b_{km}K_{ijl}{}^m = b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m + B[g_{kl}(b_{im}R_j{}^m - b_{jm}R_i{}^m) + g_{il}(b_{jm}R_k{}^m - b_{km}R_j{}^m) + g_{jl}(b_{km}R_i{}^m - b_{im}R_k{}^m)].$$

1) by theorem 4.3, if b is R -compatible then it commutes with the Ricci tensor, and K -compatibility follows.

2) if b is K -compatible it commutes with K_{ij} . Contraction with g^{kl} gives:

$$b_{im}K_j{}^m - b_{jm}K_i{}^m = (b_{im}R_j{}^m - b_{jm}R_i{}^m)[1 - B(n-2)],$$

then, if $B \neq \frac{1}{n-2}$, b commutes with the Ricci tensor and by (29) it is R -compatible. \square

Proposition 7.7. *Let K be an ABC tensor with constant $A \neq 1$ and B . If*

$$(30) \quad \nabla_i \nabla_m K_{jkl}{}^m + \nabla_j \nabla_m K_{kil}{}^m + \nabla_k \nabla_m K_{ijl}{}^m = 0$$

then the Ricci tensor is K -compatible.

Proof. If A and B are constants, one evaluates

$$(31) \quad \nabla_m K_{jkl}{}^m = (1 - A)\nabla_m R_{jkl}{}^m + \frac{1}{2}(B + 2C)(g_{kl}\nabla_j R - g_{jl}\nabla_k R),$$

Lovelock's identity (5) for the Riemann tensor implies

$$(32) \quad \nabla_i \nabla_m K_{jkl}{}^m + \nabla_j \nabla_m K_{kil}{}^m + \nabla_k \nabla_m K_{ijl}{}^m = -(1 - A)(R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m).$$

In the r.h.s. the Riemann tensor can be replaced by tensor K by (29) written for the Ricci tensor. \square

Sufficient conditions are: K is harmonic, K is recurrent (with closed recurrency parameter, see eq.(3.13) in [20]). Note that prop. 7.7 remains valid for the Weyl's conformal tensor, which is traceless.

8. WEYL-COMPATIBILITY AND GENERAL RELATIVITY

In general relativity, the Ricci tensor is related to the energy-momentum tensor by the Einstein equation: $R_{jl} = \frac{1}{2}Rg_{jl} + kT_{jl}$ with curvature $R = -2kT/(n-2)$ ($T = T^k{}_k$).

The contracted II Bianchi identity gives

$$\nabla_m R_{jkl}{}^m = k(\nabla_k T_{jl} - \nabla_j T_{kl}) + \frac{1}{2}(g_{jl}\nabla_k R - g_{kl}\nabla_j R).$$

Let K be an ABC tensor, with constant A, B, C . Its divergence (31) can be expressed in terms of the gradients of the energy momentum tensor T_{ij} . In the

same way Einstein's equations and (32) give an equation which is local in the energy momentum tensor:

$$(33) \quad \begin{aligned} & \nabla_i \nabla_m K_{jkl}{}^m + \nabla_j \nabla_m K_{kil}{}^m + \nabla_k \nabla_m K_{ijl}{}^m \\ & = -(1-A)k(T_{im}K_{jkl}{}^m + T_{jm}K_{kil}{}^m + T_{km}K_{ijl}{}^m). \end{aligned}$$

The Weyl tensor $C_{jkl}{}^m$ is the traceless part of the Riemann tensor, and it is an ABC tensor. There are advantages in discussing General Relativity by taking the Weyl tensor as the fundamental geometrical quantity [2, 15, 7]. The first equation (31)

$$\nabla_m C_{jkl}{}^m = k \frac{n-3}{n-2} \left[\nabla_k T_{jl} - \nabla_j T_{kl} + \frac{1}{n-1} (g_{jl} \nabla_k T - g_{kl} \nabla_j T) \right]$$

is reported in textbooks, as De Felice [7], Hawking Ellis [15], Stephani [31], and in the paper [2]. Instead, a further derivation yields a Bianchi-like equation for the divergence, Eq.(33), which contains no derivatives of the sources

$$(34) \quad \begin{aligned} & \nabla_i \nabla_m C_{jkl}{}^m + \nabla_j \nabla_m C_{kil}{}^m + \nabla_k \nabla_m C_{ijl}{}^m \\ & = -k \frac{n-3}{n-2} (T_{im}C_{jkl}{}^m + T_{jm}C_{kil}{}^m + T_{km}C_{ijl}{}^m). \end{aligned}$$

It can be viewed as a condition for Weyl-compatibility for the energy momentum tensor.

In view of prop.7.4 and the previous equation, the following holds:

Proposition 8.1. *If T_{ij} is Weyl-compatible, the symmetric tensor $\mathring{C}_{kl} =: T^{jm}C_{jklm}$ commutes with T_{ij} .*

In 4 dimensions, given a time-like velocity field u^i , Weyl's tensor is projected in longitudinal (electric) and transverse (magnetic) tensorial components [2]

$$E_{kl} = u^j u^m C_{jklm}, \quad H_{kl} = \frac{1}{4} u^j u^m (\epsilon_{pqjk} C^{pq}{}_{lm} + \epsilon_{pqjl} C^{pq}{}_{km})$$

that solve equations that resemble Maxwell's equations with source. Therefore, the tensor $E_{kl} = \mathring{C}_{kl}$ can be viewed as a generalized electric field. It coincides with the standard definition if $T_{ij} = (p + \rho)u_i u_j + pg_{ij}$ (perfect fluid). The generalized magnetic field is $H_{kl} = \frac{1}{4} T^{jm} (\epsilon_{pqjk} C^{pq}{}_{lm} + \epsilon_{pqjl} C^{pq}{}_{km})$.

Proposition 8.2. *If T_{kl} is Weyl compatible then $H_{kl} = 0$.*

Proof. From the condition for Weyl compatibility we obtain $\epsilon_{ijkp}[T^{im}C^{jk}{}_{lm} + T^{jm}C^{ki}{}_{lm} + T^{km}C^{ij}{}_{lm}] = 0$. The first and the second term are modified as follows:

$$\begin{aligned} \epsilon_{ijkp} T^{im} C^{jk}{}_{lm} &= \epsilon_{kijp} T^{km} C^{ij}{}_{lm} = \epsilon_{ijkp} T^{km} C^{ij}{}_{lm} \\ \epsilon_{ijkp} T^{jm} C^{ki}{}_{lm} &= \epsilon_{jkip} T^{km} C^{ij}{}_{lm} = \epsilon_{ijkp} T^{km} C^{ij}{}_{lm}. \end{aligned}$$

Then, since the sum becomes $\epsilon_{ijkp} T^{km} C^{ij}{}_{lm} = 0$, then the magnetic part of Weyl's tensor is zero. \square

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